Fun Thursday Quiz

① Favorite Book = ____ ?

② Finish the definitions.

ⓐ $\lim_{x \to a} f(x) = L$ if $\forall \varepsilon > 0$,

$\exists \delta > 0$ .....

ⓑ A set $S \subseteq \mathbb{R}$ is compact if

(i)

different

(ii)

On 4.6

$\lim_{x \to a} f(x) = L$

$\lim_{x \to a} g(x) = M$

Will need $|f(x)g(x) - LM| < \varepsilon$

$$|f(x)g(x) - Lg(x) + Lg(x) - LM| < \varepsilon$$

(4.4) $\lim_{x\to a} f(x)$ exists

$\iff \exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in \text{dom}(f)$, $0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.

$\lim_{x\to a} f(x)$ does not exist

$\iff \forall L \in \mathbb{R}, \exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists x \in \text{dom}(f)$ with $0 < |x-a| < \delta$ <u>and</u> $|f(x) - L| \geq \varepsilon$.

$\neg(A \Rightarrow B)$

A and $\neg B$

$\lim_{x\to 2} (x - \lfloor x \rfloor)^2$

<u>Alternative</u>:

$\lim_{x\to a} f(x)$ exists

$\iff \exists L \in \mathbb{R}$ s.t. $\forall$ sequences (x_n) s.t. $x_n \in \text{dom}(f)$ and $x_n \neq a$ $\forall n \in \mathbb{N}$ and $\lim x_n = a$, we have $\lim f(x_n) = L$.

$\lim_{x \to a} f(x)$ does not exist

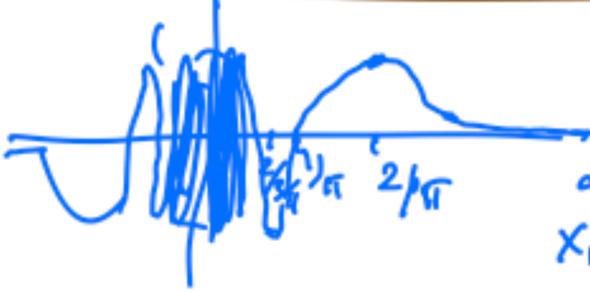
$\Longleftarrow$ $\exists\, (x_n), (y_n)$ in dom(f)
s.t. $x_n \neq a$ $\forall n$, $y_n \neq a$ $\forall n$ &

$\lim x_n = \lim y_n = a$
but $\lim f(x_n) \neq \lim f(y_n)$

$\Longleftarrow$ $\exists\, (x_n)$ in dom$(f) \setminus \{a\}$.
s.t. $\lim x_n = a$
but $\lim f(x_n)$ Does not exist.

Example: Let $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \in \mathbb{R} \setminus \{0\}$.

Prove that $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.



$x_k = \frac{2}{(4k+1)\pi}$ $f(x_k) = 1$ $\forall k$

angle $= \frac{4k+1}{2}\pi$ $y_k = \frac{1}{k\pi}$, $f(y_k) = 0$ $\forall k$.

$x_k \to 0$, $y_k \to 0$.

Pf. Consider (x_k) with $x_k = \frac{2}{(4k+1)\pi}$ for $k \in \mathbb{N}$. Then $\lim x_k = 0$ by ACT.

Consider (y_k) with $y_k = \frac{1}{k\pi}$ for $k \in \mathbb{N}$

Then $\lim y_k = 0$ by ACT.

However, $\lim f(x_k) = \lim \sin\left(\frac{(4k+1)\pi}{2}\right) = \lim 1 = 1$; and

$\lim f(y_k) = \lim \sin(k\pi) = \lim 0 = 0$.

Thus $\lim_{x \to 0} f(x)$ does not exist. ∎

Properties of Continuous functions.

If $g : A \to \mathbb{R}$ is a function, the <u>image of g</u> (<u>range of g</u>) is $g(A) = \mathrm{Im}(g) = \{g(x) : x \in A\}$.

(union of values of the function.)

Thm (The continuous image of a compact set is compact.)

i.e. If $f: A \to \mathbb{R}$ is a continuous fcn, and $C \subseteq A$ is a compact set, then $f(C)$ is also compact.

$$\left[f(C) = \{ f(x) : x \in C \}. \right].$$

Pf. With the given, let $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ be any open cover of $f(C)$.

Then $$f(C) \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha.$$

Then $$C \subseteq f^{-1}\left(\bigcup_{\alpha \in \mathcal{A}} U_\alpha \right)$$ inverse image

$$= \bigcup_{\alpha \in \mathcal{A}} f^{-1}(U_\alpha)$$ open since f is continuous

Thus, $\{f^{-1}(U_\alpha)\}_{\alpha \in \mathcal{A}}$ is an open cover of C.

C is compact $\Rightarrow$ $\exists$ finite subcover $\{f^{-1}(U_{\alpha_1}), \dots, f^{-1}(U_{\alpha_n})\}$

so that

$$C \subseteq f^{-1}(U_{\alpha_1}) \cup \dots \cup f^{-1}(U_{\alpha_n}).$$

Then

$$f(C) \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n}.$$

$\therefore$ $\{U_{\alpha_1}, \dots, U_{\alpha_n}\}$ is a finite subcover of $f(C)$.

$\therefore$ $f(C)$ is compact. $\boxtimes$

I have used facts about inverse images (which are not true for images!):

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$$

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B).$$

$$f(f^{-1}(S)) = S$$

but $f^{-1}(f(S))$ is not nec. S

$$f^{-1}(f(S)) \supseteq S \checkmark$$

Cor. (Extreme Value Theorem)

If $f : A \to \mathbb{R}$ is continuous and $C \subseteq A$ is compact. Then $f|_C$ achieves both its maximum & minimum at points of C.

i.e. $\sup\limits_{x \in C} \{f(x)\} = f(x_0)$ for some $x_0 \in C$

$\inf\limits_{x \in C} \{f(x)\} = f(x_1)$ for some $x_1 \in C$.

Remark: False if the set is not compact.

eg. $f(x) = x$ on $(0,2)$.

has no min or max on $(0,2)$.

Pf of Corollary: If C is compact then $f(C) \subseteq \mathbb{R}$ and $f(C)$ is compact. $f(C)$ is closed & bdd. $f(C)$ contains its sup & inf.! ☒

Uniform Continuity.

A function $f: A \to \mathbb{R}$ is uniformly continuous on A if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $x, c \in A$ and $|x - c| < \delta$, then

$$|f(x) - f(c)| < \varepsilon.$$

Contrast: $f: A \to \mathbb{R}$ is continuous if $\forall c \in A$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $x \in A$ and $|x-c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

Difference: for uniform continuity, δ can't depend on c.

for continuity, δ can depend on c.

Example. Prove that $f(x) = \frac{1}{x}$ is uniformly continuous on $[1, 2)$.

Scratch. Need $|f(x) - f(c)| < \varepsilon$

$1 \le c < 2$
$\quad x$

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon$$

$$\left|\frac{c-x}{xc}\right| < \varepsilon$$

$$\frac{|x-c|}{|xc|} < \varepsilon$$

$$\frac{|x-c|}{|xc|} \le \frac{|x-c|}{1} < \varepsilon \leftarrow \delta = \varepsilon$$

Pf. $\forall \varepsilon > 0$, let $\delta = \varepsilon > 0$.

Then $\forall x, c \in [1,2)$, if $|x-c| < \delta$,

then $\dfrac{|x-c|}{|xc|} \le |x-c| < \varepsilon$

$$\Rightarrow \left| \frac{x-c}{xc} \right| = \left| \overset{f(x)}{\frac{1}{x}} - \overset{f(c)}{\frac{1}{c}} \right| < \varepsilon.$$

$\therefore$ f is uniformly continuous on $[1,2)$. $\blacksquare$

Example Prove $f(x) = \frac{1}{x}$ is not unif cont. on $(0,2)$.

Scratch: I need to show $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, $\exists x, c \in (0,2)$ s.t. $|x-c| < \delta$ but $|f(x) - f(c)| \ge \varepsilon$.

Let $\varepsilon = 1$

$\frac{1}{x} - \frac{1}{c}$ $\quad \frac{2}{\delta} - \frac{1}{\delta} = \frac{1}{\delta}$.

Pf: Let $\varepsilon = 1$. Then $\forall \delta > 0$, $\exists x, c \in (0,2)$

where $x = \frac{\min(\delta, 1)}{2}$ $c = \frac{\min(\delta, 1)}{1}$

Then $|x - c| = \left|\min\{\delta, 1\}\left(\frac{1}{2} - 1\right)\right| \leq \frac{\delta}{2} < \delta$.

but $|f(x) - f(c)| = \left|\frac{1}{x} - \frac{1}{c}\right|$

$= \left|\frac{2}{\min\{\delta,1\}} - \frac{1}{\min\{\delta,1\}}\right| = \frac{1}{\min\{\delta,1\}} \geq 1 = \varepsilon$

$\therefore$ f is not unif continuous on $(0,2)$.

(But f is continuous on $(0,2)$) $\blacksquare$

Thm: A continuous fcn on a compact set is uniformly continuous.